

# Spectral analysis of time changes for horocycle flows

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## Abstract

We prove (under the condition of A. G. Kushnirenko) that all time changes for the horocycle flow have purely absolutely continuous spectrum in the orthocomplement of the constant functions. This provides an answer to a question of A. Katok and J.-P. Thouvenot on the spectral nature of time changes for horocycle flows. Our proofs rely on positive commutator methods for self-adjoint operators.

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## 1 Introduction

The purpose of this note is to provide an answer to a question of A. Katok and J.-P. Thouvenot on the spectral nature of time changes for horocycle flows.

The set-up is the standard one. Consider the unit tangent bundle  $M := T^1\Sigma$  of a compact Riemann surface  $\Sigma$  of genus  $\geq 2$ . The 3-manifold  $M$  carries a probability measure  $\mu$  which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow  $\{F_{1,t}\}_{t \in \mathbb{R}}$  and the geodesic flow  $\{F_{2,t}\}_{t \in \mathbb{R}}$ . One associates to those flows vector fields  $X_j$ , Lie derivatives  $\mathcal{L}_{X_j}$  and unitary groups  $\{U_j(t)\}_{t \in \mathbb{R}}$  in  $L^2(M, \mu)$  in the usual way. It is a classical result that the horocycle flow  $\{F_{1,t}\}_{t \in \mathbb{R}}$  is uniquely ergodic [13] and mixing of all orders [19], and that the unitary group  $\{U_1(t)\}_{t \in \mathbb{R}}$  has countable Lebesgue spectrum [21]. Furthermore, A. G. Kushnirenko [17, Thm. 2] has proved that all time changes of the horocycle flow are strongly mixing under a condition which holds if the time change is sufficiently small in the  $C^1$  topology. Namely, if  $f \in C^\infty(M)$  satisfies  $f > 0$  and  $f - \mathcal{L}_{X_2}(f) > 0$ , then the flow of the vector field  $fX_1$  is strongly mixing. This implies that the unitary group associated to  $fX_1$  has purely continuous spectrum, except at 1, where it has a simple eigenvalue.

Nothing more is known about the spectral properties of the time change  $fX_1$  (see the comments in [4, Sec. 1] and [17, Sec. 1]). However, as pointed out by A. Katok and J.-P. Thouvenot in [16, Sec. 6.3.1], it looks plausible that the unitary group associated to  $fX_1$  has purely absolutely continuous or Lebesgue spectrum. In fact, A. Katok and J.-P. Thouvenot state as a conjecture the stability of the countable Lebesgue spectrum (see [16, Conject. 6.8]). In the present note, we give an answer to the first interrogation of these authors by proving that the unitary group associated to  $fX_1$  has purely absolutely continuous spectrum outside  $\{1\}$  under the condition of A. G. Kushnirenko.

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Our proof relies on a refined version [3, 22] of a commutator method introduced by E. Mourre [20]. It uses as a starting point the well-known commutation relation satisfied by the unitary groups of the horocycle flow and the geodesic flow:

$$U_2(s)U_1(t)U_2(-s) = U_1(e^s t), \quad s, t \in \mathbb{R}. \quad (1.1)$$

To some extent, this approach has been suggested to us by the proof of A. G. Kushnirenko itself, since it already took advantage of commutator identities linking the vector fields  $X_1, X_2$  and  $fX_1$ . We also acknowledge the influence of the article [12] on commutator methods for unitary operators, and we refer to [4, 9, 10, 11, 14, 15, 23] for related works on ergodic and spectral properties of time changes. In the future, we hope that commutators methods could be used to derive spectral properties of other classes of flows than the horocycle flows considered here.

Here is a brief description of the note. In Section 2, we recall some definitions and results on positive commutator methods for self-adjoint operators. In Section 3, we introduce a generalisation of the setting presented above: We consider on an abstract (possibly noncompact)  $n$ -manifold vector fields  $X_1, X_2$  and flows  $\{F_{1,t}\}_{t \in \mathbb{R}}, \{F_{2,t}\}_{t \in \mathbb{R}}$  with unitary groups satisfying (1.1). Under an assumption generalising the one of A. G. Kushnirenko (see Assumption 3.2) we show that the self-adjoint operator associated to the time change  $fX_1$  has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue (see Theorem 3.5). We use the theory of Section 2 to prove this result. In Section 4 we apply this abstract result to the horocycle flow, taking into account the fact that the horocycle flow is strongly mixing under the condition A. G. Kushnirenko. This leads to the desired result, namely, that the unitary group associated a time change of the horocycle flow has purely absolutely continuous spectrum outside  $\{1\}$  (see Theorem 4.2).

## 2 Positive commutator methods

We recall in this section some facts on positive commutator methods borrowed from [3] and [22] (see also the original paper [20] of E. Mourre). Let  $\mathcal{H}$  be a Hilbert space with norm  $\|\cdot\|_{\mathcal{H}}$  and scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ , and denote by  $\mathcal{B}(\mathcal{H})$  the set of bounded linear operators on  $\mathcal{H}$ . Let also  $A$  be a self-adjoint operator in  $\mathcal{H}$  with domain  $\mathcal{D}(A)$ , and  $S \in \mathcal{B}(\mathcal{H})$ . For any  $k \in \mathbb{N}$ , we say that  $S$  belongs to  $C^k(A)$ , with notation  $S \in C^k(A)$ , if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H}) \quad (2.1)$$

is strongly of class  $C^k$ . In the case  $k = 1$ , one has  $S \in C^1(A)$  if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, iSA\varphi \rangle_{\mathcal{H}} - \langle A\varphi, iS\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous for the topology induced by  $\mathcal{H}$  on  $\mathcal{D}(A)$ . We denote by  $[iS, A]$  the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the function (2.1) at  $t = 0$ .

If  $H$  is a self-adjoint operator in  $\mathcal{H}$  with domain  $\mathcal{D}(H)$  and spectrum  $\sigma(H)$ , we say that  $H$  is of class  $C^k(A)$  if  $(H - z)^{-1} \in C^k(A)$  for some  $z \in \mathbb{C} \setminus \sigma(H)$ . If  $H$  is of class  $C^1(A)$ , then the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H - z)^{-1} A \varphi \rangle_{\mathcal{H}} - \langle A \varphi, (H - z)^{-1} \varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

extends continuously to a bounded form defined by the operator  $[(H - z)^{-1}, A] \in \mathcal{B}(\mathcal{H})$ . Furthermore, the set  $\mathcal{D}(H) \cap \mathcal{D}(A)$  is a core for  $H$  and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, H\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous in the topology of  $\mathcal{D}(H)$  [3, Thm. 6.2.10(b)]. This form extends uniquely to a continuous quadratic form on  $\mathcal{D}(H)$  which can be identified with a continuous operator  $[H, A]$  from  $\mathcal{D}(H)$  to the adjoint space  $\mathcal{D}(H)^*$ . In addition, the following relation holds in  $\mathcal{B}(\mathcal{H})$  :

$$[(H - z)^{-1}, A] = -(H - z)^{-1}[H, A](H - z)^{-1}. \quad (2.2)$$

Let  $E^H(\cdot)$  denote the spectral measure of the self-adjoint operator  $H$ , and assume that  $H$  is of class  $C^1(A)$ . Then, the operator  $E^H(J)[iH, A]E^H(J)$  is bounded and self-adjoint for each bounded Borel set  $J \subset \mathbb{R}$ . If there exists a number  $a > 0$  such that

$$E^H(J)[iH, A]E^H(J) \geq aE^H(J),$$

then one says that  $H$  satisfies a strict Mourre estimate on  $J$ . The main consequence of such an estimate is to imply a limiting absorption principle for  $H$  on  $J$  if  $H$  is also of class  $C^2(A)$ . This in turns implies that  $H$  has no singular spectrum in  $J$ . We recall here a version of this result valid even if  $H$  has no spectral gap (see [3, Sec. 7.1.2] and [22, Thm. 0.1] for the most general version of this result) :

**Theorem 2.1.** *Let  $H$  and  $A$  be self-adjoint operators in a Hilbert space  $\mathcal{H}$ . Suppose that  $H$  is of class  $C^2(A)$  and satisfies a strict Mourre estimate on a bounded Borel set  $J \subset \mathbb{R}$ . Then,  $H$  has no singular spectrum in  $J$ .*

### 3 Spectral analysis of time changes for abstract flows

Let  $M$  be a  $C^\infty$  manifold of dimension  $n \geq 1$  with volume form  $\Omega$ , and let  $\{F_{j,t}\}_{t \in \mathbb{R}}$ ,  $j = 1, 2$ , be (nontrivial)  $C^\infty$  complete flows on  $M$  preserving the measure  $\mu_\Omega$  induced by  $\Omega$ . Then, it is known that the operators

$$U_j(t)\varphi := \varphi \circ F_{j,t}, \quad \varphi \in C_c^\infty(M),$$

define strongly continuous unitary groups  $\{U_j(t)\}_{t \in \mathbb{R}}$  in the Hilbert space  $\mathcal{H} := L^2(M, \mu_\Omega)$  (here  $C_c^\infty(M)$  stands for the space of  $C^\infty$  functions with compact support in  $M$ ). Since  $C_c^\infty(M)$  is dense in  $\mathcal{H}$  and left invariant by  $\{U_j(t)\}_{t \in \mathbb{R}}$ , it follows from Nelson's theorem [2, Prop. 5.3] that the generator of the group  $\{U_j(t)\}_{t \in \mathbb{R}}$

$$H_j\varphi := \text{s-lim}_{t \rightarrow 0} it^{-1}\{U_j(t) - 1\}\varphi, \quad \varphi \in \mathcal{D}(H_j) := \left\{ \varphi \in \mathcal{H} \mid \lim_{t \rightarrow 0} |t|^{-1} \|\{U_j(t) - 1\}\varphi\|_{\mathcal{H}} < \infty \right\}$$

is essentially self-adjoint on  $C_c^\infty(M)$ . In fact, a direct calculation shows that

$$H_j\varphi := -i\mathcal{L}_{X_j}\varphi, \quad \varphi \in C_c^\infty(M),$$

where  $X_j$  is the (divergence-free) vector field associated to  $\{F_{j,t}\}_{t \in \mathbb{R}}$  and  $\mathcal{L}_{X_j}$  the corresponding Lie derivative. Now, suppose that there exists a  $C^1$  isomorphism  $e : (\mathbb{R}, +) \rightarrow ((0, \infty), \cdot)$  such that

$$U_2(s)U_1(t)U_2(-s) = U_1(e(s)t) \quad \text{for all } s, t \in \mathbb{R}. \quad (3.1)$$

Then, for each  $t \neq 0$ ,  $U_1(t)$  has homogeneous Lebesgue spectrum (that is, the spectrum  $\sigma(H_1)$  of  $H_1$  covers  $\mathbb{R}$ , and  $\sigma(H_1) \setminus \{0\}$  is purely Lebesgue with uniform multiplicity, see [16, Prop. 1.23]). Furthermore, if  $\mu_\Omega(M) < \infty$ , then any constant function on  $M$  is an eigenvector of  $U_1(t)$  with eigenvalue 1 (in some cases, as when the system  $(M, \mu_\Omega, F_{1,t})$  is ergodic, 1 is even a simple eigenvalue of  $U_1(t)$ ). By applying the strong derivative  $\text{id}/dt$  at  $t = 0$  in (3.1), one gets that  $U_2(s)H_1U_2(-s)\varphi = e(s)H_1\varphi$  for each  $\varphi \in C_c^\infty(M)$ . Since  $C_c^\infty(M)$  is a core for  $H_1$ , one infers that  $H_1$  is  $H_2$ -homogeneous in the sense of [7]; namely,

$$U_2(s)H_1U_2(-s) = e(s)H_1 \quad \text{on } \mathcal{D}(H_1). \quad (3.2)$$

It follows that  $H_1$  is of class  $C^\infty(H_2)$  with

$$[iH_1, H_2] = e'(0)H_1. \quad (3.3)$$

Now, consider a vector field with the same orientation and colinear to the vector field  $X_1$ , that is, a vector field  $fX_1$  where  $f \in C^\infty(M)$  satisfies  $f \geq \delta_f$  for some  $\delta_f > 0$  and  $f \in L^\infty(M)$ . The vector field  $fX_1$  has the same integral curves as  $X_1$ , but with reparametrised time coordinate. Indeed, it is known (see [15, Sec. 1] and [8, Chap. 2.2] in the compact case) that the formula

$$t = \int_0^{h(p,t)} \frac{ds}{f(F_{1,s}(p))}, \quad (p, t) \in M \times \mathbb{R},$$

defines for each  $p \in M$  a strictly increasing function  $\mathbb{R} \ni t \mapsto h(p, t) \in \mathbb{R}$  satisfying  $h(p, 0) = 0$  and  $\lim_{t \rightarrow \pm\infty} h(p, t) = \pm\infty$ . Furthermore, the implicit function theorem implies that the map  $t \mapsto h(p, t)$  is  $C^\infty$  with  $\frac{d}{dt}h(p, t) = f(F_{1,h(p,t)}(p))$ . Therefore, the function  $\mathbb{R} \ni t \mapsto \tilde{F}_{1,t}(p) \in M$  given by  $\tilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$  satisfies the initial value problem

$$\frac{d}{dt} \tilde{F}_1(p, t) = (fX_1)_{\tilde{F}_1(p,t)}, \quad \tilde{F}_1(p, 0) = p,$$

meaning that  $\{\tilde{F}_{1,t}\}_{t \in \mathbb{R}}$  is the flow of  $fX_1$ . Since the divergence  $\text{div}_{\Omega/f}(fX_1)$  of  $fX_1$  with respect to the volume form  $\Omega/f$  is zero (see [1, Prop. 2.5.23]), it follows by a standard result [1, Prop. 2.6.14] that the operator

$$\tilde{H}\varphi := -i\mathcal{L}_{fX_1}\varphi \equiv fH_1\varphi, \quad \varphi \in C_c^\infty(M),$$

is essentially self-adjoint in  $\tilde{\mathcal{H}} := L^2(M, \mu_\Omega/f)$ . Its closure is denoted by the same symbol.

In the next lemma, we introduce two auxiliary operators which will be useful for the spectral analysis of  $\tilde{H}$ .

**Lemma 3.1.** *Let  $f \in C^\infty(M)$  be such that  $f \geq \delta_f$  for some  $\delta_f > 0$  and  $f \in L^\infty(M)$ . Then,*

(a) *The operator*

$$\mathcal{U} : \mathcal{H} \rightarrow \tilde{\mathcal{H}}, \quad \varphi \mapsto f^{1/2}\varphi,$$

*is unitary with adjoint  $\mathcal{U}^* : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$  given by  $\mathcal{U}^*\psi = f^{-1/2}\psi$ .*

(b) *The symmetric operator*

$$H\varphi := f^{1/2}H_1f^{1/2}\varphi, \quad \varphi \in C_c^\infty(M),$$

*is essentially self-adjoint in  $\mathcal{H}$  (and its closure is denoted by the same symbol).*

(c) *For each  $z \in \mathbb{C} \setminus \mathbb{R}$ , the operator  $H_1 + zf^{-1}$  is invertible with bounded inverse, and satisfies*

$$(H + z)^{-1} = f^{-1/2}(H_1 + zf^{-1})^{-1}f^{-1/2}. \quad (3.4)$$

*Proof.* Point (a) follows from a direct calculation taking into account the boundedness of  $f$  from below and from above. For (b), observe that

$$H\varphi = f^{-1/2}fH_1f^{1/2}\varphi = \mathcal{U}^*\tilde{H}\mathcal{U}\varphi$$

for each  $\varphi \in \mathcal{U}^*C_c^\infty(M)$ . So,  $H$  is essentially self-adjoint on  $\mathcal{U}^*C_c^\infty(M) \equiv C_c^\infty(M)$ . To prove (c), take  $z \equiv \lambda + i\mu \in \mathbb{C} \setminus \mathbb{R}$ ,  $\varphi \in \mathcal{D}(H_1 + zf^{-1}) \equiv \mathcal{D}(H_1)$  and  $\{\varphi_n\} \subset C_c^\infty(M)$  such that  $\lim_n \|\varphi - \varphi_n\|_{\mathcal{D}(H_1)} = 0$ . Then, it follows from (b) that

$$\|(H_1 + zf^{-1})\varphi\|_{\mathcal{H}}^2 = \lim_n \|f^{-1/2}(H + z)f^{-1/2}\varphi_n\|_{\mathcal{H}}^2 \geq \inf_{p \in M} f^{-2}(p)\mu^2 \|\varphi\|_{\mathcal{H}}^2,$$

and thus  $H_1 + z f^{-1}$  is invertible with bounded inverse (see [2, Lemma 3.1]). Now, to show (3.4), take  $\psi = (H + z)\zeta$  with  $\zeta \in C_c^\infty(M)$ , observe that

$$(H + z)^{-1}\psi - f^{-1/2}(H_1 + z f^{-1})^{-1}f^{-1/2}\psi = 0, \quad (3.5)$$

and then use the density of  $(H + z)C_c^\infty(M)$  in  $\mathcal{H}$  to extend the identity (3.5) to all of  $\mathcal{H}$ .  $\square$

The proof of Lemma 3.1(b) implies that  $H$  and  $\tilde{H}$  are unitarily equivalent. Therefore, one can either work with  $H$  in  $\mathcal{H}$  or with  $\tilde{H}$  in  $\tilde{\mathcal{H}}$  to determine the spectral properties associated with the time change  $fX_1$ . For convenience, we present in the sequel our results for the operator  $H$ . We start by collecting all the necessary assumptions on the function  $f$  (note that the assumption  $f \in C^\infty(M)$  is made essentially for convenience; if need be, the results of this note can certainly be extended to the case  $f \in C^2(M)$ ).

**Assumption 3.2** (Time change). *The function  $f \in C^\infty(M)$  is such that*

- (i)  $f \geq \delta_f$  for some  $\delta_f > 0$ ,
- (ii) the functions  $f, \mathcal{L}_{X_1}(f), \mathcal{L}_{X_2}(f), \mathcal{L}_{X_1}(\mathcal{L}_{X_2}(f))$  and  $\mathcal{L}_{X_2}(\mathcal{L}_{X_2}(f))$  belong to  $L^\infty(M)$ ,
- (iii) the function  $g := \frac{e'(0)f - \mathcal{L}_{X_2}(f)}{2f}$  satisfies  $g \geq \delta_g$  for some  $\delta_g > 0$ .

If  $M$  is compact, then (ii) is automatically verified and (i) and (iii) are satisfied if  $f$  and  $e'(0)f - \mathcal{L}_{X_2}(f)$  are strictly positive functions. Therefore, Assumption 3.2 reduces to the assumptions of A. G. Kushnirenko [17, Thm. 2].

In the next lemma, we prove regularity properties of  $H$  and  $H^2$  with respect to  $H_2$  which will be useful when deriving the strict Mourre estimate.

**Lemma 3.3.** *Let  $f$  satisfy Assumption 3.2, take  $\alpha \in \{\pm 1/2, \pm 1\}$  and let  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then,*

- (a) the multiplication operators  $g^\alpha$  and  $f^\alpha$  satisfy  $g^\alpha, f^\alpha \in C^1(H_2)$  and  $g^\alpha \in C^1(H)$  with
$$[ig^\alpha, H_2] = -\alpha g^{\alpha-1} \mathcal{L}_{X_2}(g), \quad [if^\alpha, H_2] = -\alpha f^{\alpha-1} \mathcal{L}_{X_2}(f) \quad \text{and} \quad [ig^\alpha, H] = -\alpha f g^{\alpha-1} \mathcal{L}_{X_1}(g),$$
- (b)  $(H + z)^{-1} \in C^1(H_2)$  with  $[i(H + z)^{-1}, H_2] = -(H + z)^{-1}(Hg + gH)(H + z)^{-1}$ ,
- (c)  $(H^2 + 1)^{-1} \in C^1(H_2)$  with  $[i(H^2 + 1)^{-1}, H_2] = -(H^2 + 1)^{-1}(H^2g + 2HgH + gH^2)(H^2 + 1)^{-1}$ ,
- (d)  $(H^2 + 1)^{-1} \in C^2(H_2)$ .

*Proof.* (a) Simple computations using the linearity of  $\mathcal{L}_{X_2}$  and the bound  $f \geq \delta_f$  imply that

$$\mathcal{L}_{X_2}(f^{1/2}) = \frac{1}{2} f^{-1/2} \mathcal{L}_{X_2}(f).$$

Thus, one has for each  $\varphi \in C_c^\infty(M)$

$$\langle H_2 \varphi, f^{1/2} \varphi \rangle_{\mathcal{H}} - \langle \varphi, f^{1/2} H_2 \varphi \rangle_{\mathcal{H}} = \langle \varphi, [H_2, f^{1/2}] \varphi \rangle_{\mathcal{H}} = \langle \varphi, -\frac{i}{2} f^{-1/2} \mathcal{L}_{X_2}(f) \varphi \rangle_{\mathcal{H}}.$$

Since  $f^{-1/2} \mathcal{L}_{X_2}(f) \in L^\infty(M)$ , it follows by the density of  $C_c^\infty(M)$  in  $\mathcal{D}(H_2)$ , that  $f^{1/2} \in C^1(H_2)$  with  $[H_2, f^{1/2}] = -\frac{i}{2} f^{-1/2} \mathcal{L}_{X_2}(f)$ . The other identities can be shown similarly.

(b) Let  $t \in \mathbb{R}$  and  $\varphi \in \mathcal{H}$ . Then, one infers from Equations (3.2) and (3.4) that

$$\begin{aligned} & e^{-itH_2}(H + z)^{-1} e^{itH_2} \varphi \\ &= e^{-itH_2} f^{-1/2} e^{itH_2} (e(t)H_1 + z e^{-itH_2} f^{-1} e^{itH_2})^{-1} e^{-itH_2} f^{-1/2} e^{itH_2} \varphi. \end{aligned}$$

So, one gets from point (a), Equation (3.4) and Lemma 3.1(b) that

$$\begin{aligned}
& \frac{d}{dt} e^{-itH_2} (H+z)^{-1} e^{itH_2} \varphi \Big|_{t=0} \\
&= [if^{-1/2}, H_2] (H_1 + zf^{-1})^{-1} f^{-1/2} \varphi + f^{-1/2} (H_1 + zf^{-1})^{-1} [if^{-1/2}, H_2] \varphi \\
&\quad - f^{-1/2} (H_1 + zf^{-1})^{-1} \{e'(0)H_1 + z[if^{-1}, H_2]\} (H_1 + zf^{-1})^{-1} f^{-1/2} \varphi \\
&= \frac{1}{2} f^{-1} \mathcal{L}_{X_2}(f) (H+z)^{-1} \varphi + \frac{1}{2} (H+z)^{-1} f^{-1} \mathcal{L}_{X_2}(f) \varphi \\
&\quad - (H+z)^{-1} \{e'(0)H + zf^{-1} \mathcal{L}_{X_2}(f)\} (H+z)^{-1} \varphi \\
&= \frac{1}{2} (H+z)^{-1} H f^{-1} \mathcal{L}_{X_2}(f) (H+z)^{-1} \varphi + \frac{1}{2} (H+z)^{-1} f^{-1} \mathcal{L}_{X_2}(f) H (H+z)^{-1} \varphi \\
&\quad - (H+z)^{-1} e'(0) H (H+z)^{-1} \varphi \\
&= -(H+z)^{-1} (Hg + gH) (H+z)^{-1} \varphi,
\end{aligned}$$

which implies the claim.

(c) Let  $\varphi \in \mathcal{H}$ . Then, it follows from point (b) that

$$\begin{aligned}
& \frac{d}{dt} e^{-itH_2} (H^2 + 1)^{-1} e^{itH_2} \varphi \Big|_{t=0} \\
&= \frac{d}{dt} e^{-itH_2} (H+i)^{-1} e^{itH_2} e^{-itH_2} (H-i)^{-1} e^{itH_2} \varphi \Big|_{t=0} \\
&= -(H+i)^{-1} (Hg + gH) (H+i)^{-1} (H-i)^{-1} \varphi - (H+i)^{-1} (H-i)^{-1} (Hg + gH) (H-i)^{-1} \varphi \\
&= -(H^2 + 1)^{-1} (H^2 g + 2HgH + gH^2) (H^2 + 1)^{-1} \varphi,
\end{aligned}$$

which implies the claim.

(d) Direct computations using point (c) show that

$$\begin{aligned}
[i(H^2 + 1)^{-1}, H_2] &= -(H^2 + 1)^{-1} \{ (H^2 + 1)g - 2g + g(H^2 + 1) \\
&\quad + 2(H+i)g(H-i) - 2ig(H-i) + 2i(H+i)g \} (H^2 + 1)^{-1} \\
&= -2 \operatorname{Re} \{ g(H^2 + 1)^{-1} - (H^2 + 1)^{-1} g(H^2 + 1)^{-1} \\
&\quad + (H-i)^{-1} g(H+i)^{-1} + 2i(H-i)^{-1} g(H^2 + 1)^{-1} \}.
\end{aligned}$$

Therefore, the claim readily follows from the fact that the operators  $g, (H^2 + 1)^{-1}, (H-i)^{-1}$  and  $(H+i)^{-1}$  belong to  $C^1(H_2)$ .  $\square$

In order to apply the theory of Section 2, one has to prove at some point a positive commutator estimate. Usually, one proves it for the operator  $H$  under study. But in our case, the commutator  $[iH, H_2] = Hg + gH$  appearing in Lemma 3.3(b) (which is the simplest nontrivial commutator in our set-up) does not exhibit any explicit positivity. By contrast, the commutator  $[iH^2, H_2] = H^2g + 2HgH + gH^2$  of Lemma 3.3(c) is made of the positive operators  $g, H^2$  and  $HgH$ , and thus  $[iH^2, H_2]$  is more likely to be positive as a whole. The formalisation of this intuition is the content of the next lemma.

**Lemma 3.4** (Strict Mourre estimate for  $H^2$ ). *Let  $f$  satisfy Assumption 3.2 and let  $J$  be a bounded Borel set in  $(0, \infty)$ . Then,*

$$E^{H^2}(J) [iH^2, H_2] E^{H^2}(J) \geq a E^{H^2}(J)$$

with  $a := 2\delta_g \cdot \inf(J) > 0$ .

*Proof.* We know from Equation (2.2) and Lemma 3.3(c) that

$$E^{H^2}(J)[iH^2, H_2]E^{H^2}(J) = E^{H^2}(J)(H^2g + 2HgH + gH^2)E^{H^2}(J).$$

We also know from Assumption 3.2(iii) that

$$E^{H^2}(J)2HgHE^{H^2}(J) \geq aE^{H^2}(J)$$

with  $a = 2\delta_g \cdot \inf(J) > 0$ . Therefore, it is sufficient to show that  $E^{H^2}(J)(H^2g + gH^2)E^{H^2}(J) \geq 0$ .

So, for any  $\varepsilon > 0$  let  $H_\varepsilon^2 := H^2(\varepsilon^2 H^2 + 1)^{-1}$  and  $H_\varepsilon^\pm := H(\varepsilon H \pm i)^{-1}$ . Then, the inclusion  $g^{1/2} \in C^1(H)$  of Lemma 3.3(a) implies that

$$\text{s-lim}_{\varepsilon \searrow 0} [H_\varepsilon^\pm, g^{1/2}] = \pm \text{s-lim}_{\varepsilon \searrow 0} (\varepsilon H \pm i)^{-1} [iH, g^{1/2}] (\varepsilon H \pm i)^{-1} = \pm i [g^{1/2}, H].$$

Therefore, for each  $\varphi \in \mathcal{H}$  it follows that

$$\begin{aligned} & \langle \varphi, E^{H^2}(J)(H^2g + gH^2)E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \lim_{\varepsilon \searrow 0} \langle \varphi, E^{H^2}(J)(H_\varepsilon^2 g^{1/2} g^{1/2} + g^{1/2} g^{1/2} H_\varepsilon^2)E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \lim_{\varepsilon \searrow 0} \langle \varphi, E^{H^2}(J)([H_\varepsilon^2, g^{1/2}]g^{1/2} + 2g^{1/2}H_\varepsilon^2g^{1/2} + g^{1/2}[g^{1/2}, H_\varepsilon^2])E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &\geq \lim_{\varepsilon \searrow 0} \langle \varphi, E^{H^2}(J)([H_\varepsilon^2, g^{1/2}]g^{1/2} + g^{1/2}[g^{1/2}, H_\varepsilon^2])E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \lim_{\varepsilon \searrow 0} \langle \varphi, E^{H^2}(J)(H_\varepsilon^+ [H_\varepsilon^-, g^{1/2}]g^{1/2} + [H_\varepsilon^+, g^{1/2}]H_\varepsilon^- g^{1/2} \\ &\quad + g^{1/2}[g^{1/2}, H_\varepsilon^+]H_\varepsilon^- + g^{1/2}H_\varepsilon^+[g^{1/2}, H_\varepsilon^-])E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \lim_{\varepsilon \searrow 0} \langle \varphi, E^{H^2}(J)(H[H, g^{1/2}]g^{1/2} + [H_\varepsilon^+, g^{1/2}]g^{1/2}H_\varepsilon^- + [H_\varepsilon^+, g^{1/2}][H_\varepsilon^-, g^{1/2}] \\ &\quad + g^{1/2}[g^{1/2}, H]H + H_\varepsilon^+ g^{1/2}[g^{1/2}, H_\varepsilon^-] + [g^{1/2}, H_\varepsilon^+][g^{1/2}, H_\varepsilon^-])E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \langle \varphi, E^{H^2}(J)(H[H, g^{1/2}]g^{1/2} + [H, g^{1/2}]g^{1/2}H + 2[H, g^{1/2}]^2 + g^{1/2}[g^{1/2}, H]H \\ &\quad + Hg^{1/2}[g^{1/2}, H])E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &= \langle \varphi, E^{H^2}(J)2[H, g^{1/2}]^2 E^{H^2}(J)\varphi \rangle_{\mathcal{H}} \\ &\geq 0, \end{aligned}$$

which implies the claim.  $\square$

Using the previous results for  $H^2$ , one can finally determine spectral properties of  $H$  :

**Theorem 3.5** (Spectral properties of  $H$ ). *Let  $f$  satisfy Assumption 3.2. Then,  $H$  has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue.*

*Proof.* We know from Lemmas 3.3(d) and 3.4 that  $(H^2 + 1)^{-1} \in C^2(H_2)$  and that  $H^2$  satisfies a strict Mourre estimate on each bounded Borel subset of  $(0, \infty)$ . It follows by Theorem 2.1 that  $H^2$  has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue. Accordingly, the Hilbert space  $\mathcal{H}$  admits the orthogonal decomposition

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\text{ac}}(H^2),$$

with  $\mathcal{H}_{\text{ac}}(H^2)$  the subspace of absolute continuity of  $H^2$ .

Now, the function  $\lambda \mapsto \lambda^2$  has the Luzin N property on  $\mathbb{R}$ ; namely, if  $J$  is a Borel subset of  $\mathbb{R}$  with Lebesgue measure zero, then  $J^2$  also has Lebesgue measure zero. It follows that  $\mathcal{H}_{\text{ac}}(H^2) \subset \mathcal{H}_{\text{ac}}(H)$ ,

with  $\mathcal{H}_{\text{ac}}(H)$  the subspace of absolute continuity of  $H$  (see Proposition 29, Section 3.5.4 of [5]). Since  $\ker(A^2) = \ker(A)$  for all self-adjoint operators  $A$ , we thus infer that

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\text{ac}}(H^2) \subset \ker(H) \oplus \mathcal{H}_{\text{ac}}(H).$$

So, one necessarily has  $\mathcal{H} = \ker(H) \oplus \mathcal{H}_{\text{ac}}(H)$ , meaning that  $H$  has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue.  $\square$

## 4 Spectral analysis of time changes for horocycle flows

In this section, we apply the results of Section 3 to time changes for horocycle flows on compact surfaces of constant negative curvature.

Let  $\Sigma$  be a compact Riemann surface of genus  $\geq 2$  and let  $M := T^1\Sigma$  be the unit tangent bundle of  $\Sigma$ . The 3-manifold  $M$  carries a probability measure  $\mu_\Omega$  (induced by a canonical volume form  $\Omega$ ) which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow  $\{F_{1,t}\}_{t \in \mathbb{R}}$  and the geodesic flow  $\{F_{2,t}\}_{t \in \mathbb{R}}$ . Both flows correspond to right translations on  $M$  when  $M$  is identified with a homogeneous space  $\Gamma \backslash \text{PSL}(2; \mathbb{R})$ , for some cocompact lattice  $\Gamma$  in  $\text{PSL}(2; \mathbb{R})$  (see [6, Sec. III.3 & Sec. IV.1] for details). We denote by  $\{U_1(t)\}_{t \in \mathbb{R}}$  and  $\{U_2(t)\}_{t \in \mathbb{R}}$  the corresponding unitary groups in  $\mathcal{H} := L^2(M, \mu_\Omega)$ , and we write  $X_j$  (resp.  $H_j$ ) for the vector field (resp. self-adjoint generator) associated to  $\{U_j(t)\}_{t \in \mathbb{R}}$ ,  $j = 1, 2$  (see Section 3). It is a classical result that the horocycle flow  $\{F_{1,t}\}_{t \in \mathbb{R}}$  is uniquely ergodic [13] and mixing of all orders [19], and that  $U_1(t)$  has countable Lebesgue spectrum for each  $t \neq 0$  (see [16, Prop. 2.2] and [21]). Moreover, the identity (3.1) holds with  $e : \mathbb{R} \rightarrow (0, \infty)$  the exponential, *i.e.*

$$U_2(s)U_1(t)U_2(-s) = U_1(e^s t) \quad \text{for all } s, t \in \mathbb{R}$$

(here we consider the negative horocycle flow  $\{F_{1,t}\}_{t \in \mathbb{R}} \equiv \{F_{1,t}^-\}_{t \in \mathbb{R}}$ , but everything we say can be adapted to the positive horocycle flow by inverting a sign, see [6, Rem. IV.1.2]).

Now, consider a time change  $fX_1$  of  $X_1$  with  $f \in C^\infty(M)$  satisfying Assumption 3.2, let  $H$  be the self-adjoint operator as in Lemma 3.1(b), and let  $\tilde{H}$  be the self-adjoint operator associated to  $fX_1$ . Since  $M$  is compact, Assumption 3.2 reduces to the following:

**Assumption 4.1.** *The functions  $f \in C^\infty(M)$  and  $f - \mathcal{L}_{X_2}(f) \in C^\infty(M)$  are strictly positive.*

Under Assumption 4.1, A. G. Kushnirenko [17, Thm. 2] has shown that the flow generated by the vector field  $fX_1$  is strongly mixing (see [18, Sec. 4] for a generalisation of this result). So,  $\tilde{H}$  has purely continuous spectrum, except at 0, where it has a simple eigenvalue (see *e.g.* [6, Sec. I.2]). Moreover, the flows  $\{F_{1,t}\}_{t \in \mathbb{R}}$ ,  $\{F_{2,t}\}_{t \in \mathbb{R}}$  and the function  $f$  satisfy all the assumptions of Section 3. Therefore, Theorem 3.5 implies that  $H$  has no singular continuous spectrum. These properties, together with the fact that  $H$  and  $\tilde{H}$  are unitarily equivalent, lead to the following result:

**Theorem 4.2.** *Let  $f$  satisfy Assumption 4.1. Then, the self-adjoint operator  $\tilde{H}$  associated to the vector field  $fX_1$  has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.*

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